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# *FWI*- ideals of Residuated Lattice Wajsberg Algebras

**A. Ibrahim*[1] and R. Shanmugapriya[2]**

[1]Assistant Professor,
P.G.and Research Department of Mathematics,
H.H.The Rajah's College, Pudukkotai-622001, Tamilnadu, INDIA.
[2]Department of Mathematics,
Avinashilingam Institute for Home Science and Higher Education for Women,
Coimbatore-641043,Tamilnadu, INDIA.
email:[1]ibrahimaadhil@yahoo.com; [1]dribrahimaadhil@gmail.com; [2]priyasanmu@gmail.com

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**ABSTRACT**

In this paper, we introduce the notion of fuzzy Wajsberg implicative ideal (*FWI*-ideal) of residuated lattice Wajsberg algebra and investigate some properties with illustrations. Further, we show that every *FWI* – ideal of residuated lattice Wajsberg algebra is a fuzzy lattice ideal.

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## 1. INTRODUCTION

Fuzzy logic is a form of many-valued logic in which the truth values of variables may be any real number between 0 and 1. The term fuzzy logic was introduced with the 1965 proposal of fuzzy set theory by Loffi Zadeh[11]. Idea of Zadeh's have been applied in the field of algebraic structures, the study of fuzzy algebras has achieved great success. Mordchaj Wajsberg[8] proposed the concept of Wajsberg algebras in 1935. Rose and Rosser[7] published the proof of Wajsberg algebras in 1958.In 1984, Font, Rodriguez and Torrens[3] extended Wajsberg algebras as an alternative model for the infinite valued Lukasiewicz logic and introduced lattice structure of Wajsberg algebra. Lattice Wajsberg algebra provide the foundation to establish the corresponding logic system in algebraic view point. Further, they[3] introduced the notions of implicative filters and family of implicative filters in a lattice

Wajsberg algebras and investigated their properties. Fuzzy subset has been applied to the theories of filters and ideals in various non-classical logical algebras. Basheer Ahamed and Ibrahim[1, 2] introduced the definitions of fuzzy implicative and an anti fuzzy implicative filters of lattice Wajsberg algebras. Ibrahim and Shajitha Begum[4, 5] introduced the notions of *WI*-ideal, fuzzy and normal fuzzy *WI*-ideal of lattice Wajsberg algebra and discussed some related properties. Recently, the authors[6] introduced the notion of Wajsberg implicative ideal(*WI*-ideal) of residuated lattice Wajsberg algebra and discussed some properties.

The aim of this paper is to introduce the notion of fuzzy Wajsberg implicative ideal (*FWI*-ideal) of residuated lattice Wajsberg algebras and discuss some properties with examples. Also, we show that every fuzzy Wajsberg implicative ideal (*FWI*-ideal) of residuated lattice Wajsberg algebra is a fuzzy lattice ideal.

## 2. PRELIMINARIES

In this section, we recall some basic definitions and properties which are helpful to develop our main results.

**Definition2.1[3].** Let $(A,\rightarrow,*,1)$ be an algebra with a binary operation "$\rightarrow$" and a quasi complement "$*$" is called a Wajsberg algebra if and only if it satisfies the following axioms for all $x,y,z \in A$,

(i) $1 \rightarrow x = x$

(ii) $(x \rightarrow y) \rightarrow y = ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$

(iii) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$

(iv) $(x^* \rightarrow y^*) \rightarrow (y \rightarrow x) = 1.$

**Definition 2.2[3].** AWajsberg algebra $(A,\rightarrow,*,1)$ satisfies the following axioms for all $x,y,z \in A$,

(i) $x \rightarrow x = 1$

(ii) If $(x \rightarrow y) = (y \rightarrow x) = 1$ then $x = y$

(iii) $x \rightarrow 1 = 1$

(iv) $(x \rightarrow (y \rightarrow x)) = 1$

(v) If $(x \rightarrow y) = (y \rightarrow z) = 1$ then $x \rightarrow z = 1$

(vi) $(x \rightarrow y) \rightarrow \big((z \rightarrow x) \rightarrow (z \rightarrow y)\big) = 1$

(vii) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$

(viii) $x \rightarrow 0 = x \rightarrow 1^* = x^*$

(ix) $(x^*)^* = x$

(x) $(x^* \rightarrow y^*) = y \rightarrow x.$

**Definition 2.3[3].** A Wajsberg algebra $A$ is called a lattice Wajsberg algebra, if it satisfies the following conditions for all $x,y \in A$,

(i) The partial ordering "$\leq$" on a lattice Wajsberg algebra, such that $x \leq y$ if and only if $x \rightarrow y = 1$

(ii) $x \vee y = (x \to y) \to y$

(iii) $x \wedge y = ((x^* \to y^*) \to y^*)^*$.

Thus, $(A, \vee, \wedge, *, 0, 1)$ is a lattice Wajsberg algebra with lower bound 0 and upper bound 1.

**Proposition 2.4[3].** A Wajsberg algebra $(A, \to, *, 1)$ satisfies the following axioms for all $x, y, z \in A$,

(i) If $x \leq y$ then $x \to z \geq y \to z$ and $z \to x \leq z \to y$

(ii) $x \leq y \to z$ if and only $if$ $y \leq x \to z$

(iii) $(x \vee y)^* = (x^* \vee y^*)$

(iv) $(x \wedge y)^* = (x^* \vee y^*)$

(v) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$

(vi) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$

(vii) $(x \to y) \vee (y \to x) = 1$

(viii) $x \to (y \vee z) = (x \to y) \vee (x \to z)$

(ix) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$

(x) $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$

(xi) $(x \wedge y) \to z = (x \to y) \to (x \to z)$.

**Definition 2.5[9].** A residuated lattice $(A, \vee, \wedge, \otimes, \to, 0, 1)$ satisfies the following conditions for all $x, y, z \in A$,

(i) $(A, \vee, \wedge, 0, 1)$ is a bounded lattice,

(ii) $(A, \otimes, 1)$ is commutative monoid,

(iii) $x \otimes$ y $\leq z$ if and only if $x \leq y \to z$.

**Definition 2.6[9].** In a residuated lattice $(A, \vee, \wedge, \otimes, \to, 0, 1)$ the binary operation " $\oplus$ " is defined by $x \oplus$ y $= x^* \to y$ for all $x, y \in A$.

**Definition 2.7[4].** Let $(A, \vee, \wedge, *, \to, 1)$ be a lattice wajsberg algebra. If a binary operation " $\otimes$ " on $A$ satisfies $x \otimes y = (x \to y^*)^*$ for all $x, y \in A$. Then $(A, \vee, \wedge, \otimes, \to, *, 0, 1)$ is called a residuated lattice Wajsberg algebra.

**Definition 2.8[4].** Let $A$ be a lattice Wajsberg algebra. Let $I$ be a nonempty subset of $A$, then $I$ is called a *WI*-ideal of lattice Wajsberg algebra $A$ satisfies for all $x, y \in A$,

(i) $0 \in I$

(ii) $(x \to y)^* \in I$ and $y \in I$ imply $x \in I$ .

**Definition 2.9[4].** Let $L$ be a lattice. An ideal $I$ of $L$ is a nonempty subset of $L$ is called a lattice ideal, if it satisfies the following axioms for all $x, y \in A$,

(i) $x \in I, y \in L$ and $y \leq x$ imply $y \in I$

(ii) $x, y \in I$ implies $x \vee y \in I$.

**Definition 2.10[6].** Let $A$ be a residuated lattice Wajsberg algebra. Let $I$ be a nonempty subset of $A$, then $I$ is called a $WI$-ideal of residuated lattice Wajsberg algebra $A$ satisfies the following axioms for all $x, y \in A$,

(i) $0 \in I$
(ii) $x \otimes y \in I$ and $y \in I$ imply $x \in I$
(iii) $(x \to y)^* \in I$ and $y \in I$ imply $x \in I$.

**Definition 2.11[6].** Let $A$ be a residuated lattice Wajsberg algebra. A non empty subset $I$ of $A$ is called a *WI* -ideal of $A$ if $I$ satisfies for all $x, y \in A$,
(i) If $x \leq y$ and $y \in I$, then $x \in I$
(ii) $x \oplus y \in I$.

**Definition 2.12[5].**Let $A$ be a set. A function $\mu: A \to [0,1]$ is called a fuzzy subset on $A$ for each $x \in A$, the value of $\mu(x)$ describes a degree of membership of $x$in $\mu$.

**Definition 2.13[5].** Let $A$ be a lattice Wajsberg algebra. A fuzzy subset $\mu$ of $A$ is called a fuzzy *WI*-ideal of $A$ if it satisfies for all $x, y \in A$,
(i) $\mu(0) \geq \mu(x)$
(ii) $\mu(x) \geq \min\{\mu\big((x \to y)^*\big), \mu(y)\big\}$.

**Definition 2.14[5].**A fuzzy subset$\mu$ of a lattice Wajsberg algebra$A$ is called a fuzzy lattice ideal if it satisfies for all $x, y \in A$,
(i) If $y \leq x$ then $\mu(y) \geq \mu(x)$
(ii) $\mu(x \vee y) \geq \min\{\mu(x), \mu(y)\}$.

**Definition 2.15[10].**Let $\mu$ be a fuzzy subset of a residuated lattice $A. \mu$ is called a fuzzy ideal of $A$, if $\mu$ satisfies the following conditions for all $x, y \in A$,
(i) If $x \leq y$, then $\mu(x) \geq \mu(y)$
(ii) $\mu(x \oplus y) \geq \min\{\mu(x), \mu(y)\}$.

## 3. FUZZY WAJSBERG IMPLICATIVE IDEAL (*FWI*-ideal)

In this section, we definea *FWI*-ideal of a residuated latticeWajsberg algebra and obtain some useful results with illustrations.

**Definition3.1**.Let $A$ be a residuated latticeWajsberg algebra. A fuzzy subset $\mu$ of $A$ is called a Fuzzy Wajsberg implicative Ideal (*FWI*- ideal) of residuated lattice Wajsberg algebra $A$ if for all $x, y \in A$,
(i) $\mu(0) \geq \mu(x)$
(ii) $\mu(x) \geq \min\{\mu(x \otimes y), \mu(y)\}$
(iii) $\mu(x) \geq \min\{\mu\big((x \to y)^*\big), \mu(y)\big\}$.

**Proposition 3.2.**onsider a set $A = \{0, a, b, c, d, 1\}$. Define a partial ordering "$\leq$" on $A$, such that $0 \leq a \leq b \leq c \leq d \leq 1$ with a binary operations"$\otimes$"and "$\to$"and a quasi complement "$*$"on $A$ as in the following tables 3.1 and 3.2.

**Table 3.1: Complement**

| $x$ | $x^*$ |
|---|---|
| 0 | 1 |
| $a$ | $c$ |
| $b$ | $d$ |
| $c$ | $a$ |
| $d$ | $b$ |
| 1 | 0 |

**Table 3.2: Implication**

| $\to$ | 0 | $a$ | $b$ | $c$ | $d$ | 1 |
|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| $a$ | $c$ | 1 | $b$ | $c$ | $b$ | 1 |
| $b$ | $d$ | $a$ | 1 | b | $a$ | 1 |
| $c$ | $a$ | $a$ | 1 | 1 | $a$ | 1 |
| $d$ | $b$ | 1 | 1 | $b$ | 1 | 1 |
| 1 | 0 | $a$ | $b$ | $c$ | $d$ | 1 |

Define $\vee$ and $\wedge$ operations on $A$ as follows:

$$(x \vee y) = (x \to y) \to y,$$
$$(x \wedge y) = (x^* \to y^*) \to y^*)^*; \;\; x \otimes y = (x \to y^*)^* \text{ for all } x, y \in A.$$

Then, $A$ is a residuated lattice Wajsberg algebra.

Consider the fuzzy subset $\mu$ on $A$ is defined by $\mu(x) = \begin{cases} 0.8 & \text{if } x = 0 \text{ for all } x \in A \\ 0.2 & \text{otherwise for all } x \in A \end{cases}$

Then, $\mu$ is a *FWI* -ideal of $A$.

**Example 3.3.** Consider a set $A$ ={$0, a, b, c, 1$}. Define a partial ordering "$\leq$" on $A$ such that $0 \leq a \leq b \leq c \leq 1$ with a binary operations"$\otimes$"and "$\to$"and a quasi complement "$*$"on $A$ as in the following tables 3.3 and 3.4.

**Table 3.3: Complement**

| $x$ | $x^*$ |
|---|---|
| 0 | 1 |
| $a$ | $a$ |
| $b$ | $c$ |
| $c$ | $b$ |
| 1 | 0 |

**Table 3.4: Implication**

| $\to$ | 0 | $a$ | $b$ | $c$ | 1 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 |
| $a$ | $a$ | $a$ | $c$ | 1 | 1 |
| $b$ | $c$ | 1 | 1 | 1 | 1 |
| $c$ | $b$ | 1 | 1 | 1 | 1 |
| 1 | 0 | $a$ | $b$ | $c$ | 1 |

Define $\vee$ and $\wedge$ operations on $A$ as follows:

$$(x \vee y) = (x \to y) \to y,$$
$$(x \wedge y) = (x^* \to y^*) \to y^*)^*; \;\; x \otimes y = (x \to y^*)^* \text{for all } x, y \in A$$

Then, $A$ is a residuated lattice Wajsberg algebra.

Consider the fuzzy subset $\mu$ on $A$ is defined by $\mu(x) = \begin{cases} 0.7 & \text{if } x = 0 \text{ for all } x \in A \\ 0.3 & \text{otherwise for all } x \in A \end{cases}$

Then, $\mu$ is a *FWI* - ideal of $A$.

**Example 3.4.** Consider a set $A$ ={$0, a, b, c, 1$}. Define a partial ordering "$\leq$" on $A$ such that $0 \leq a \leq b \leq c \leq 1$ with a binary operations"$\otimes$"and "$\to$"and a quasi complement "$*$"on $A$ as in the following tables 3.5 and 3.6.

**Table 3.5: Complement**

| $x$ | $x^*$ |
|---|---|
| 0 | 1 |
| $a$ | $c$ |
| $b$ | $b$ |
| $c$ | $a$ |
| 1 | 0 |

**Table 3.6: Implication**

| $\to$ | 0 | $a$ | $b$ | $c$ | 1 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 |
| $a$ | $c$ | 1 | 1 | 1 | 1 |
| $b$ | $b$ | $c$ | 1 | 1 | 1 |
| $c$ | $a$ | $b$ | 1 | 1 | 1 |
| 1 | 0 | $a$ | $b$ | $c$ | 1 |

Define $\vee$ and $\wedge$ operations on $A$ as follows:

$$(x \vee y) = (x \to y) \to y,$$
$$(x \wedge y) = (x^* \to y^*) \to y^*)^*; \ \ x \otimes y = (x \to y^*)^* \text{for all } x, y \in A$$

Then, $A$ is a residuated lattice Wajsberg algebra.

Consider the fuzzy subset $\mu$ on $A$ is defined by $\mu(x) = \begin{cases} 0.7 & \text{if } x \in \{0, b\} \text{for all } x \in A \\ 0.3 & \text{if } x \in \{a, c, 1\} \text{for all } x \in A \end{cases}$

Then, we get $\mu$ is not a *FWI* -ideal of residuated lattice Wajsberg algebra $A$. Since, we have $\mu(c) = 0.3$. But $\min\{\mu(c \otimes b), \mu(b)\} = 0.7$.

**Example 3.5.**Let $A$ be a residuated lattice Wajsberg algebra defined in Example 3.3, fuzzy subset $\mu$ of $A$ is defined by $\mu(0) = \mu(c)$ and $\mu(0) \geq \mu(x)$ for all $x \in \{a, b, 1\}$, then $\mu$ is a *FWI* -ideal of $A$.

**Proposition 3.6.** Every *FWI*-ideal $\mu$ of a residuated lattice Wajsberg algebra $A$ is order reversing.

***Proof.*** Let $\mu$ be a *FWI*-ideal of $A$ and let $y \leq x$, then$(y \to x)^* = 1^* = 0, (y \to x^*)^* = (y \to y)^* = 1^* = 0$ for all $x, y \in A$.

Now, $\mu(y) \geq \min\{\mu(y \otimes x), \mu(x)\}$ [From (ii) of Definition 3.1]

$= \min\{\mu(y \to x^*)^*, \mu(x)\}$ [From Definition 2.7]

$= \min\{\mu(0), \mu(x)\} = \mu(x)$. [From (i) of Definition 3.1]

And $\mu(y) \geq \min\{\mu(y \to x)^*, \mu(x)\}$

$= \min\{\mu(0), \mu(x)\} = \mu(x)$. [From (i) of Definition 3.1]

Thus, we have $\mu(y) \geq \mu(x)$.

This shows that $\mu$ is order reversing.■

**Proposition 3.7.** Let $A$ be a residuated lattice Wajsberg algebra. Every *FWI*-ideal of $A$ is a fuzzy lattice ideal.

***Proof.*** Let $\mu$ be a *FWI*-ideal of $A$.

Then $\mu(y) \geq \mu(x)$ for all $x, y \in A$ [From Definition (i) of 3.1 and Proposition 3.6]

**To prove :** $\mu(x \vee y) \geq \min\{\mu(x), \mu(y)\}$

Now, we have $\big((x \vee y) \otimes y\big) = \big((x \vee y) \to y^*)\big)^*$ [From Definition 2.7]

$= ((x \to y) \to y) \to y^*)^*)$ [From (ii) of Definition 2.3]

$= (1 \to x)^* = x^*$ [From (i) of Definition 2.1]

$= y$ for all $x, y \in A$

Now, $\mu((x \vee y) \otimes y) \geq \min\{\mu((x \vee y) \otimes y), \mu(y)\}$

$= \min\{\mu(y), \mu(y)\}$
$= \mu(y)$ for all $x, y \in A$

Also, $((x \vee y) \rightarrow y)^* = ((x \rightarrow y) \rightarrow y) \rightarrow y)^*$ [From (ii) of Definition 2.3]
$\leq (x^*)^* = x$ for all $x, y \in A$ [From (ix) of Definition 2.2]

Thus, we have $\mu(x \vee y) \geq \min\{\mu((x \vee y) \rightarrow y)^*, \mu(y)\}$ [From (ii) of Definition 2.13]
$= \min\{\mu(x), \mu(y)\}$
$= \mu(y)$ for all $x, y \in A$

Therefore, $\mu$ is a fuzzy lattice ideal. ■

**Note.** The converse of Proposition 3.7 is not true.
It is easy to verify from example 3.2 $\mu$ is a fuzzy subset of $A$ defined by

$$(x) = \begin{cases} 0.9 & \text{if } x \in \{0, c, d\} \text{ for all } x \in A \\ 0.2 & \text{if } x \in \{a, b, 1\} \text{ for all } x \in A \end{cases}$$

Then, $\mu$ is a fuzzy residuated lattice ideal of $A$ but not *FWI*-ideal. Since, we have $\mu(a) = 0.2$. But $\min\{\mu(a \otimes d), \mu(d)\} = 0.9$.

**Proposition 3.8.** Let $\mu$ be a *FWI*-ideal of $A$. The following hold for all $x, y \in A$

(i) $\mu(x \vee y) = \min\{\mu(x), \mu(y)\}$
(ii) $\mu(x \wedge y) \geq \min\{\mu(x), \mu(y)\}$
(iii) $\mu(x \otimes y) \geq \min\{\mu(x), \mu(y)\}$
(iv) $\mu(x \oplus y) = \min\{\mu(x), \mu(y)\}$

***Proof*.** Let $\mu$ be a *FWI*-ideal of $A$.
Now, we have $x \otimes y \leq x \wedge y \leq x \vee y \leq x \oplus y$ for all $x, y \in A$.
Then, $\mu(x \otimes y) \geq \mu(x \wedge y) \geq \mu(x \vee y) \geq \mu(x \oplus y) \geq \min\{\mu(x), \mu(y)\}$.
[From (ii) of Definition 2.15]
Since $x \oplus y \geq x \vee y \geq x, y$
It follows that $\mu(x \oplus y) \leq \mu(x \vee y) \leq \min\{\mu(x), \mu(y)\}$
Thus, we have $\mu(x \oplus y) \leq \mu(x \vee y) \leq \min\{\mu(x), \mu(y)\}$ for all $x, y \in A$. ■

**Proposition 3.9.** Let $\mu$ be a fuzzy subset of a residuated latticeWajsbergalgebera$A$. Then $\mu$ is a *FWI*-ideal of $A$ if and only if the level set $\mu_t (\neq \varphi)$ is a *WI*-ideal of $A$.
***Proof*.** Let $\mu$ be a *FWI*-ideal of $A$ and $\mu_t \neq \varphi$.
If $x \leq y$ and $y \in \mu_t$ we have $\mu(y) \geq t$, for all $x, y \in A$.
Again we have $x \leq y$, It follows that $\mu(x) \geq \mu(y) \geq t$
Then, we have $x \in \mu_t$ . [From (i) of Definition 2.11]
Now, let $x, y \in \mu_t$ we have $\mu(x) \geq t$ and $\mu(y) \geq t$,
Then, $\mu(x \oplus y) \geq \min\{\mu(x), \mu(y)\}$ [From (ii) of Definition 2.15]
$\geq t$
Hence, we have $x \oplus y \in \mu_t$ [From (ii) of Definition 2.11]
Therefore, $\mu_t$ is a *WI*-ideal of$A$.

Conversely, assume that $\mu_t$ is a *FWI*-ideal of $A$.
Let $t = \min\{\mu(x), \mu(y)\}$, then $x \in \mu_t$ and $y \in \mu_t$ for all $x, y \in A$.
By $\mu_t$ is a *WI*-ideal, we get $x \oplus y \in \mu_t$ [From (ii) of Defintion 2.11]
and $\mu(x \oplus y) \geq \min\{\mu(x), \mu(y)\}$ [From (ii) of Definition 2.15]
$= t$
Let $x \leq y$ and $t = \mu(y)$, then $y \in \mu_t$ for all $x, y \in A$.
Hence $x \in \mu_t$ [From (i) of Definition 2.11]
Which implies $\mu(x) \geq \mu(y)$ for all $x, y \in A$. [From (i) of Definition 2.15]
Therefore, $\mu$ is a *FWI*-ideal of $A$. ■

**Proposition 3.10.** Let $\mu$ be a fuzzy a subset of a residuated latticeWajsberg algebra $A$. $\mu$ is a *FWI*-ideal of $A$, if $\mu$ satisfies the following conditions for all $x, y \in A$,
(i) $\mu(0) \geq \mu(x)$
(ii) $\mu(y) \geq \min\{\mu(x), \mu(x^* \otimes y)\}$.
***Proof*.** Let $\mu$ be a *FWI*-ideal of $A$.
If $0 \leq x$, it follows that $\mu(0) \geq \mu(x)$ for all $x, y \in A$ [From (i) of Definition 2.15]
Now, we have $x \oplus (x^* \otimes y) = x^* \to (x^* \to y)$ [From (i) of Definition 2.6]
$\geq y$ for all $x, y \in A$
And, we have $\mu(y) \geq \mu\big(x \oplus (x^* \otimes y)\big)$ [From Proposition 3.8]
$\geq \min\{\mu(x),\ \mu(x^* \otimes y)\}$ for all $x, y \in A$ [From (ii) of Definition 2.15]
Conversely, assume that $\mu$ satisfied by (i) and (ii)
**To prove:** (i) $\mu(x) \geq \mu(y)$; (ii) $\mu(x \oplus y) \geq \min\{\mu(x), \mu(y)\}$
Let $x \leq y$ then $y^* \leq x^*$ for all $x, y \in A$
It follows that $x \otimes y^* \leq x \otimes x^* = 0$
We have $\mu(0) = \mu(y^* \otimes x)$.
From (ii), we have $\mu(x) \geq \min\{\mu(y), \mu(y^* \otimes x)\} = \min\{\mu(y), \mu(0)\} \geq \mu(y)$.
Thus, $\mu(x) \geq \mu(y)$ for all $x, y \in A$ [From Definition (i) 2.15]
$x, y \in A$,
Now, we have $x^* \otimes (x \oplus y) = x^* \otimes (x^* \to y) \leq y$,
Then, we have $\mu(x^* \otimes (x \oplus y) \geq \mu(y)$ for all $x, y \in A$
And $\mu(x \oplus y) \geq \min\{\mu(x), \mu\big(x^* \otimes (x \oplus y)\big)\} \geq \min\{\mu(x), \mu(y)\}$
Hence $\mu(x \oplus y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in A$ [From Definition (ii) of 2.15]
Therefore, $\mu$ is a *FWI*-ideal of $A$. ■

**Proposition 3.11.** Let $\mu$ be a fuzzy subset of a residuated lattice $A$. $\mu$ is a *FWI*-ideal of $A$, if $\mu$ satisfies the following conditions for all $x, y \in A$,
(i) $\mu(0) \geq \mu(x)$
(ii) $\mu(y) \geq \min\{\mu(x), \mu(x^* \to y^*)^*\}$.
***Proof.*** Let $\mu$ be a *FWI*-idealof $A$.
Let $0 \leq x$ for any $x \in A$, it follows that $\mu(0) \geq \mu(x)$. [From Definition 2.15(i)]
Thus, (i) is satisfied.

Since, $x^* \otimes y^{**} \leq (x^* \otimes y^{**})^{**} = (x^* \rightarrow y^*)^*$ [From (ix) of Definition 2.2]
We have $\mu((x^* \rightarrow y^*)^*) \leq \mu(x^* \otimes y^{**})$.
It follows that,

$$\mu(y^{**}) \geq \mu(x), \mu(x^* \otimes y^{**}) \geq \min\{\mu(x), \mu((x^* \otimes \mathrm{y})\} \geq \min\{\mu(x), \mu((x^* \rightarrow y^*)^*)\}.$$

Since$y^{**} \geq y$, we have $\mu(y) \geq \mu(y^{**})$.
Thus, we have $\mu(y) \geq \min\{\mu(x), \mu((x^* \rightarrow y^*)^*)\}$ for all $x, y \in A$
Thus, (ii) is satisfied.
Conversely, assume that $\mu$ satisfies (i) and (ii)
**To prove:**$\mu$ is a *FWI*-ideal
In (ii), taking $y = x^{**}$, we have $\mu(x^{**}) \geq \mu(x)$for all $x, y \in A$
Now, $x^* \otimes y^{**} \leq (x^* \otimes y^{**})^{**} = (x^* \rightarrow y^*)^*$for all $x, y \in A$
We have $\mu(y) \geq \min\{\mu(x), \mu((x^* \rightarrow y^*)^*)\} \geq \min\{\mu(x), \mu(x^* \otimes \mathrm{y})\}$.
Therefore, $\mu$ is a *FWI*-ideal of $A$. ■

**Remark 3.12.**Let $\mu$ be a *FWI*-ideal of a residuated lattice $A$. Then, $\mu(x^{**}) = \mu(x)$ for all $x \in A$.

**Proposition 3.13.**Let $\mu$ be a fuzzy subset of a residuated lattice $A$.$\mu$ is a *FWI*-idealof $A$, if $\mu$ satisfies the following conditions for all $x, y \in A$,

(i) $\mu(x \oplus y) \geq \min\{\mu(x), \mu(y)\}$
(ii) $\mu(x \wedge y) \geq \mu(x)$.

***Proof.*** Let $\mu$ be a *FWI*-idealof $A$.
Now, we have $x \wedge y \leq x$, then, we have $\mu(x \wedge y) \geq \mu(x)$ for all $x, y \in A$
Conversely, assume that $\mu$ satisfies (i) and (ii)
**To prove:**$\mu(x \wedge y) \geq \min\{\mu(x), \mu(y)\}$
Let $y \leq x$, then $x \wedge y = y$ and $\mu(y) = \mu(x \wedge y) \geq \mu(x)$ for all $x, y \in A$
Hence, $\mu$ is a *FWI*-idealof $A$. ■
Next, we define a fuzzy subset which is useful in next proposition.

**Definition3.14**. Let $I$ be a nonempty subset of $A$ and $\alpha, \beta \in [0,1]$such that $\alpha > \beta$. We define subset $\mu_I$ by $\mu_I(x) = \begin{cases} \alpha, & \text{if } x \in I, \\ \beta, & \text{otherwise.} \end{cases}$

**Proposition 3.15.**Let $I$ be a non-empty subset of $A$. Then $\mu_I$ is a *FWI*-ideal of $A$ if and only if $I$ is a *WI*-ideal of$A$.
***Proof.*** Let $\mu_I$be a *FWI*-ideal of $A$.
Then, $\mu_I(x) = \mu_I(y)$= $\alpha$ for all $x, y \in A$.
Now, we have $\mu_I(x \oplus y) \geq \min\{\mu_I(x), \mu_I(y)\} = \alpha$ [From (ii) of Definition 2.15]
And $x \oplus y \in I$. [From (i) of Definition 2.11]
Let $x \leq y$ and $y \in I$ then $\mu_I(x) \geq \mu_I(y)$ and
$\mu_I(y) = \alpha$ for all $x, y \in A$ [From (i) of Definition 2.15]
Hence, $\mu_I(x) = \alpha$, that is $x \in I$. [From (ii) of Definition 2.11]

Therefore, $I$ is a *WI*-ideal of$A$.
Conversely, Let $I$ be a *WI*-ideal of $A$
**To prove:** $\mu_I$ is a $FWI$-ideal

**Case I:** If $x, y \in I$ then $x \oplus y \in I$.
Thus, we have

$$\mu_I(x \oplus y) = \alpha = \min\{\mu_I(x), \mu_I(y)\} \text{ for all } x, y \in A \text{ [From (ii) of Definition 2.15]}$$

**Case II:** If $x \notin$ For $y \notin$ F. Then, $\mu_I(x) = \beta$ or $\mu_I(y) = \beta$.
Thus, we have $\mu_I(x \oplus y) \geq \beta = \min\{\mu_I(x), \mu_I(y)\}$. [From (ii) of Definition 2.15]
From case I to case II, we have

$$\mu_I(x \oplus y) \geq \min\{\mu_I(x), \mu_I(y)\} \text{forall } x, y \in A \text{ [From (ii) of Definition 2.15]}$$

Let $x \leq y$ then
(case I) If $y \in I$ then $x \in I$.Thus, we have $\mu_I(y) = \alpha = \mu_I(x)$ for all $x, y \in A$
(case II) If $y \notin$ I then $\mu_I(y) = \beta$.
Thus, we have $\mu_I(x) \geq \mu_I(y) = \beta$.
Hence, $x \leq y$, we have $\mu_I(x) \geq \mu_I(y)$ for all $x, y \in A$ [From (i) of definition 2.15]
Therefore, $\mu_I$ is a *FWI*-ideal. ■

**Proposition 3.16.** Let $\mu$ be a *FWI*-ideal of a residuated lattice $A$. Then the set $I_0 = \{x \in A | \mu(x) = \mu(0)\}$is a *FWI*-ideal of $A$.

***Proof.*** Let $x, y \in I_0$,then $\mu(x) = \mu(y) = \mu(0)$and
We have $\mu(x \oplus y) \geq \min\{\mu(x), \mu(y)\}$ [From (ii) of Definition 2.15]
$= \mu(0)$for all $x, y \in A$
Since, $\mu(0) \geq \mu(x)$for all $x \in A$ [From (i) of Definition 3.1]
We have $\mu(0) \geq \mu(x \oplus y)$for all $x, y \in A$
Hence, $\mu(0) = \mu(x \oplus y)$, that is, $x \oplus y \in I_0$.
Let $x \leq y$ and $y \in I_0$ then $\mu(x) \geq \mu(y)$ [From (i) of Definition 2.15]
$= \mu(0)$for all $x, y \in A$
We have $x \in I_0$. [From (i) of Defintion 2.11]
Therefore, $I_0$ is a *WI*-ideal of $A$. ■

## 4. CONCLUSION

In this paper, we have introduced the notion of fuzzy Wajsberg implicative ideal (*FWI*-ideal) of residuated lattice Wajsberg algebra and discussed some of their properties with examples. Further, we have shown that every *FWI*-ideal of residuated lattice Wajsberg algebra is a fuzzy lattice ideal.

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