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On Hyper-Zagreb Index of Three Graph Operations

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ABSTRACT

The Hyper-Zagreb index of a graph $G = (V, E)$ is defined as
$HZ(G) = \sum_{ab\epsilon E}[d_G(a) + d_G(b)]^2$,
where $d_G(u)$ is the degree of the vertex u in G. In this paper, some explicit expressions of the Hyper-Zagreb index and its coindices of the tensor product of *n* graphs, strong product of graphs and semi-strong product of graphs are derived.

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1. INTRODUCTION

Chemical compounds are often modeled by graph structures, where atoms are denoted by vertices and their molecular bonding are represented by edges. Such simple graphs are known as *Molecular graphs*. According to IUPAC, a *topological index (TI)* is a numerical value associated with a graph structure, which has correlation with various physical and chemical properties, or biological activities. In last few decades, thousands of topological indices are defined and studied not only as “molecular descriptors” but also as “graph invariants”, because these values are preserved under graph automorphism. So researchers from various disciplines like chemical sciences, mathematical sciences, information sciences and engineering come forward to study the mathematical properties as well as the applications of these topological indices in QSAR and QSPR. Another important aspect of this kind of study is that it can reduce the cost of laboratory experiments, which may not be always possible due to constraints of fund and time.

The TIs are defined based on various parameters of a graph like degree, distance, eigenvalues or even edges in the complement graph. Degree-based topological indices are the most popular ones, which draw tremendous attention. The first genuine attempt to define a degree-based topological index was by Randić in 1975 [15]. An account of some popular degree-based indices and their comparison is reported by Gutman[8]. Then in 1991, Graovać *et al*., have computed Wiener's index[22] of the Cartesian product of graphs[6], which is a distance based topological index. This can be considered to be the first of such attempts to compute any topological index of the product of graphs. A relatively new inclusion to this wide list of indices is Hyper-Zagreb index (HZI), which was put forward by Shirdel *et al*., in 2013 [18]. The HZI and also its coindices are well studied in last few years [1,2,3,5,10,14,17,18,20]. In [18], HZI of the Cartesian product, composition, join and disjunction of two graphs are computed. Basavanagoud *et al*., have derived the HZI of the Corona product of graphs[1], and of four new sums of graphs[2], proposed by Eliasi and Taeri[4]. They have also corrected some errors present in[18]. The HZI of Cartesian product and join of n graphs, splice and link, chains are reported in[10]. In[14, 17, 20] results of HZI and its coindices of various graphs and graph operations like double, extended double, C_4nanotube, rectangular grid, prism, complete n-partite graph, vertex corona, edge corona and Mycielskian are presented.

In this paper, we have derived the HZI and its coindices of the tensor product of *n* graphs, strong product of graphs and semi-strong products of graphs. Throughout the paper we consider simple connected graphs only. The rest of the paper is organized as follows. In section 2, we include some preliminaries. Main results are presented in section 3. As application, some examples are also included in this section. Conclusions are made in section 4.

2. PRELIMINARIES

Throughout the following discussion we denote the degree of a vertex v in a graph G by $d_G(v)$ or simply by $d(v)$ and we call two vertices u and v in a graph G as adjacent vertices if there is an edge connecting them and we denote it by $u \sim v$ and if u is not adjacent to v then we denote it by $u \nsim v$. In this section, we formally present the definitions of some topological indices and the three graph operations under consideration.

First Zagreb Index: The first Zagreb index of a graph G is defined as

$M_1(G) = \sum_{v \epsilon V(G)} d^2(v) = \sum_{x \sim y}[d(x) + d(y)]$

First General Zagreb Index: The first general Zagreb index is also studied, which can be defined for any $\alpha \geq 1$ as

$M_1^\alpha(G) = \sum_{v \epsilon V(G)} d^\alpha(v) = \sum_{x \sim y}[d^{\alpha-1}(x) + d^{\alpha-1}(y)]$

Obviously, the parameter α is usually required to be different from 0 and 1 only. But the interesting case can be with $\alpha \geq 1$. The study of the cases $\alpha < 0$ and $0 < \alpha < 1$ is analogous, *mutatis mutandis*.

Second Zagreb Index: The second Zagreb index of a graph G is defined as

$M_2(G) = \sum_{u \sim v} d(u)d(v)$

Some results on first and second Zagreb indices of various products of graphs may be found in[11, 19, 23].

Hyper Zagreb Index (HZI): The HZI of a graph G is defined as
$HZ(G) = \sum_{a\sim b}[d(a) + d(b)]^2$.

Hyper Zagreb Coindex: The Hyper-Zagreb coindex of a graph G is defined as
$\overline{HZ}(G) = \sum_{a\nsim b}[d(a) + d(b)]^2$.

Tensor Product: The tensor product of two graphs G_1 and G_2 is denoted by $G_1 \otimes G_2$ and is defined as the graph whose vertex set is $V(G_1) \times V(G_2)$ and any two vertices (a, x) and (b, y) in $G_1 \otimes G_2$ are adjacent if ab is an edge in G_1 and xy is an edge in G_2. The tensor product of graphs also known as *Kronecker product* of graphs was first proposed by *Weichsel* [21]. From this definition it is clear that, $d_{G_1 \otimes G_2}(a, x) = d(a)d(x)$, and the number of edges in $G_1 \otimes G_2$ is $2|E(G_1)||E(G_2)|$.

Strong Product: The strong product of two graphs G_1 and G_2 is denoted by $G_1 \boxtimes G_2$ and is defined as the graph whose vertex set is $V(G_1) \times V(G_2)$ and any two vertices (a, x) and (b, y) in $G_1 \boxtimes G_2$ are adjacent if $\{ab \in E(G_1)\ and\ xy \in E(G_2)\}$ *or* $\{a = b\ and\ xy \in E(G_2)\}$ *or* $\{ab \in E(G_1)\ and\ x = y\}$. It was proposed by *Sabidussi* in 1960 [16]. From this definition it is clear that, $d_{G_1 \boxtimes G_2}(a, x) = d(a) + d(x) + d(a)d(x)$, and
$|E(G_1 \boxtimes G_2)| = 2|E(G_1)||E(G_2)| + |V(G_1)||E(G_2)| + |V(G_2)||E(G_1)|$.

Semi Strong Product: The semi strong product of two graphs G_1 and G_2 is the graph $G_1 \overline{\times} G_2$ with vertex set $V(G_1 \overline{\times} G_2) = V(G_1) \times V(G_2)$ and edge set $E(G_1 \overline{\times} G_2) = \{(a, x)(b, y) | [ab \in E(G_1)\ and\ x = y]$ or $[ab \in E(G_1)\ and\ xy \in E(G_2)]\}$. From this definition it is clear that, $d_{G_1 \overline{\times} G_2}(a, x) = d(a)[d(x) + 1]$ and the number of edges in $G_1 \overline{\times} G_2$ is $2|E(G_1)||E(G_2)| + |V(G_2)||E(G_1)|$.

The detail of these products of graphs can be found in[9, 13, 19, 23]. Few recent contributions related to semi-strong product of graphs may be found in[12, 24].

3. MAIN RESULTS

Lemma 3.1: Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with $|V_1| = n_1, |V_2| = n_2, |E_1| = m_1,\ and\ |E_2| = m_2.$ Then, $M_1(G_1 \boxtimes G_2) = (n_2 + 4m_2)M_1(G_1) + (n_1 + 4m_1)M_1(G_2) + M_1(G_1)M_1(G_2) + 8m_1m_2$.

Proof: For the proof of this lemma we refer to *Theorem 2.6* of [19]■

Lemma 3.2: Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs, with $|V_1| = n_1, |V_2| = n_2, |E_1| = m_1, |E_2| = m_2$. Then, $M_1(G_1 \overline{\times} G_2) = M_1(G_1)[M_1(G_2) + 4m_2 + n_2]$.

Proof:

$$\begin{aligned} M_1(G_1 \overline{\times} G_2) &= \sum_{(u,v)\in V(G_1 \overline{\times} G_2)} d^2(u, v) \\ &= \sum_{(u,v)\in V(G_1 \overline{\times} G_2)} d^2(u)[d^2(v) + 2d(v) + 1] \\ &= \sum_{u\in V(G_1)} d^2(u) \sum_{v\in V(G_2)}[d^2(v) + 2d(v) + 1] \\ &= M_1(G_1)[M_1(G_2) + 4m_2 + n_2] \blacksquare \end{aligned}$$

Theorem 3.3: Let G_1 and G_2 be any two graphs. Then

$HZ(G_1 \otimes G_2) = HZ(G_1)HZ(G_2) - [2HZ(G_1)M_2(G_2) + 2HZ(G_2)M_2(G_1)] + 8M_2(G_1)M_2(G_2)$

Proof: Let $a, b \in V(G_1)$, and $x, y \in V(G_2)$. We have,

$$HZ(G_1 \otimes G_2) = \sum_{(a,x)\sim(b,y)}[d(a,x) + d(b,y)]^2$$
$$= \sum_{(a,x)\sim(b,y)}[d^2(a,x) + d^2(b,y)] + 2\sum_{(a,x)\sim(b,y)} d(a,x)d(b,y)$$
$$= \sum_{a\sim b}\sum_{x\sim y} d^2(a)d^2(x) + d^2(b)d^2(y) + \sum_{a\sim b}\sum_{x\sim y} d^2(a)d^2(y) + d^2(b)d^2(x) + 2.2\sum_{a\sim b}\sum_{x\sim y} d(a)d(b)d(x)d(y)$$
$$= \sum_{a\sim b}\sum_{x\sim y}[d^2(a)d^2(x) + d^2(b)d^2(y) + d^2(a)d^2(y) + d^2(b)d^2(x) + 4d(a)d(b)d(x)d(y)]$$
$$= \sum_{a\sim b}\sum_{x\sim y}[d^2(a)d^2(x) + d^2(b)d^2(y) + d^2(a)d^2(y) + d^2(b)d^2(x) + 2d^2(a)d(x)d(y) + 2d^2(b)d(x)d(y) + 2d(a)d(b)d^2(x) + 2d(a)d(b)d^2(y) + 4d(a)d(b)d(x)d(y) - 2\{d^2(a)d(x)d(y) + d^2(b)d(x)d(y) + 2d(a)d(b)d(x)d(y) + d(a)d(b)d^2(x) + d(a)d(b)d^2(y) + 2d(a)d(b)d(x)d(y)\} + 8d(a)d(b)d(x)d(y)]$$
$$= \sum_{a\sim b}[d^2(a) + d^2(b) + 2d(a)d(b)]\sum_{x\sim y}[d^2(x) + d^2(y) + 2d(x)d(y)] - 2\sum_{a\sim b}[d^2(a) + d^2(b) + 2d(a)d(b)]\sum_{x\sim y} d(x)d(y) - 2\sum_{x\sim y}[d^2(x) + d^2(y) + 2d(x)d(y)]\sum_{a\sim b} d(a)d(b) + 8\sum_{a\sim b} d(a)d(b)\sum_{x\sim y} d(x)d(y)$$
$$= HZ(G_1)HZ(G_2) - [2HZ(G_1)M_2(G_2) + 2HZ(G_2)M_2(G_1)] + 8M_2(G_1)M_2(G_2),$$

which completes the proof ■

Theorem 3.4: Let $G_1, G_2, \dots, G_n$ be n graphs. Then

$$HZ(G_1 \otimes G_2 \otimes \dots \otimes G_n) = \prod_{i=1}^{n} HZ(G_i) - [2\sum_{i=1}^{n} M_2(G_i)\prod_{j\neq i} HZ(G_j) - 2^2\sum_{i,j} M_2(G_i)M_2(G_j)\prod_{k\neq i,j} HZ(G_k) + \dots + (-1)^{n-1}2^n\prod_{i=1}^{n} M_2(G_i)] + 2^n\prod_{i=1}^{n} M_2(G_i) \quad \dots\dots (*)$$

Proof: We prove this by induction. Using *Theorem 3.3*, clearly the expression $(*)$ is true for product of two graphs. So, we have

$$HZ(G_1 \otimes G_2 \otimes \dots \otimes G_n) = HZ((G_1 \otimes G_2 \otimes \dots \otimes G_{n-1}) \otimes G_n)$$
$$= HZ(G_1 \otimes G_2 \otimes \dots \otimes G_{n-1})HZ(G_n) - 2M_2(G_1 \otimes G_2 \otimes \dots \otimes G_{n-1})HZ(G_n) - 2M_2(G_n)HZ(G_1 \otimes G_2 \otimes \dots \otimes G_{n-1}) + 8M_2(G_1 \otimes G_2 \otimes \dots \otimes G_{n-1})M_2(G_n) \quad \dots\dots(3.4.1)$$

Let $(*)$ is true for product of $n-1$ graphs. So, we have

$$HZ(G_1 \otimes G_2 \otimes \dots \otimes G_{n-1}) = \prod_{i=1}^{n-1} HZ(G_i) - [2\sum_{i=1}^{n-1} M_2(G_i)\prod_{j\neq i} HZ(G_j) - 2^2\sum_{i,j} M_2(G_i)M_2(G_j)\prod_{k\neq i,j} HZ(G_k) + \dots + (-1)^{n-2}2^{n-1}\prod_{i=1}^{n-1} M_2(G_i)] + 2^{n-1}\prod_{i=1}^{n-1} M_2(G_i)$$

Therefore,

$$HZ(G_1 \otimes G_2 \otimes \dots \otimes G_{n-1})HZ(G_n) = \prod_{i=1}^{n} HZ(G_i) - [2\sum_{i=1}^{n-1} M_2(G_i)\prod_{i\neq j}^{n} HZ(G_j) - 2^2\sum_{i,j} M_2(G_i)M_2(G_j)\prod_{k\neq i,j}^{n} HZ(G_k) + \dots + (-1)^{n-2} 2^{n-1}\prod_{i=1}^{n-1} M_2(G_i)\, HZ(G_n)] + 2^{n-1}\prod_{i=1}^{n-1} M_2(G_i)\, HZ(G_n) \quad \dots\dots(3.4.2)$$

Again, since it can be easily shown that, $M_2(G_1 \otimes G_2 \otimes \dots \otimes G_n) = 2^{n-1}\sum_{i=1}^{n} M_2(G_i)$, therefore

$$2M_2(G_1 \otimes G_2 \otimes G_3 \dots \otimes G_{n-1})HZ(G_n) = 2.\,2^{n-2}\prod_{i=1}^{n-1} M_2(G_i)\, HZ(G_n)$$

$= 2^{n-1}\prod_{i=1}^{n-1} M_2(G_i)\, HZ(G_n)$ ……(3.4.3)

$2M_2(G_n)HZ(G_1 \otimes G_2 \otimes \dots \otimes G_{n-1}) = 2\prod_{i=1}^{n-1} HZ(G_i)\, M_2(G_n) -$
$[2^2 \sum_{i=1}^{n-1} M_2(G_i) \prod_{j\neq i}^{n-1} HZ(G_j)\, M_2(G_n) \quad -2^3 \sum_{i,j} M_2(G_i) M_2(G_j)\, M_2(G_n) \prod_{k\neq i,j}^{n-1} HZ(G_k) +$
$\dots + (-1)^{n-2} 2^n \prod_{i=1}^{n} M_2(G_i)] + 2^n \prod_{i=1}^{n} M_2(G_i)$ ……(3.4.4)

$8M_2(G_1 \otimes G_2 \otimes \dots \otimes G_{n-1}) M_2(G_n) = 2^{n+1} \prod_{i=1}^{n} M_2(G_i)$ ……(3.4.5)

Using $(3.4.2) - (3.4.5)$ in $(3.4.1)$, the desired result can be obtained ■

The following corollary follows as a direct consequence of the above theorem.

Corollary 3.5: If $G_1 = G_2 = \dots = G_n = G$ then
$HZ(\otimes^n G) = [HZ(G)]^n - [2\binom{n}{1} M_2(G)[HZ(G)]^n - 2^2\binom{n}{2}[M_2(G)]^2[HZ(G)]^{n-2} + \dots +$
$2^{n-1}\binom{n}{n-1}[M_2(G)]^{n-1}HZ(G) + (-1)^{n-1}2^n[M_2(G)]^n] + 2^n[M_2(G)]^n$

Theorem 3.6: Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs, with $|V_1| = n_1, |V_2| = n_2, |E_1| = m_1, |E_2| = m_2$. Then
$HZ(G_1 \boxtimes G_2) = HZ(G_1)HZ(G_2) + 3[M_1(G_2)HZ(G_1) + M_1(G_1)HZ(G_2)] - 2[M_2(G_1)HZ(G_2) +$
$M_2(G_2)HZ(G_1)] + 12[m_2 M_1(G_1) + m_1 M_1(G_2)] + 6[m_2 HZ(G_1) + m_1 HZ(G_2)] +$
$8M_2(G_1)M_2(G_2) + n_2 HZ(G_1) + n_1 HZ(G_2)$

Proof: Let $a, b, u \in V_1$, and $x, y, v \in V_2$. We have,

$HZ(G_1 \boxtimes G_2) = \sum_{(a,x)\sim(b,y)} [d(a,x) + d(b,y)]^2$
$= \sum_{(a,x)\sim(b,y)} [d^2(a.x) + d^2(b,y)] + 2\sum_{(a,x)\sim(b,y)} d(a,x)d(b,y)$
$= \sum_{(u,v)\in V_1\times V_2} d^3(u,v) + 2\sum_{(a,x)\sim(b,y)} d(a,x)d(b,y)$ ……(3.6.1)

Now,

$\sum_{(u,v)\in V_1\times V_2} d^3(u,v) = \sum_{u\in V_1}\sum_{v\in V_2} [d(u) + d(v) + d(u)d(v)]^3$
$= \sum_{u\in V_1}\sum_{v\in V_2} [d^3(u) + d^3(v) + d^3(u)d^3(v) + 3d^2(u)d(v) + 3d(u)d^2(v)$
$+3d^3(u)d(v) + 3d(u)d^3(v) + 3d^3(u)d^2(v) + 3d^2(u)d^3(v) + 6d^2(u)d^2(v)]$ ….(3.6.2)

Again,

$\sum_{u\in V_1}\sum_{v\in V_2} d^3(u) = n_2 \sum_{u\in V_1} d^3(u) = n_2 \sum_{pq\in E_1} [d^2(p) + d^2(q)]$
$= n_2 \sum_{pq\in E_1} [d(p) + d(q)]^2 - 2d(p)d(q) = n_2 HZ(G_1) - 2n_2 M_2(G_1)$

Similarly, $\sum_{u\in V_1}\sum_{v\in V_2} d^3(v) = n_1 HZ(G_2) - 2n_1 M_2(G_2)$. Also, we know that
$\sum_{u\in V_1}\sum_{v\in V_2} d^i(u) d^j(v) = \sum_{u\in V_1} d^i(u) \sum_{u\in V_2} d^i(v)$
$= \sum_{pq\in E_1} [d^{i-1}(p) + d^{i-1}(q)] \sum_{rs\in E_2} [d^{j-1}(r) + d^{j-1}(s)]$, where $i, j \geq 2$

Hence,

$\sum_{u\in V_1}\sum_{v\in V_2} d^3(u)d^3(v) = [HZ(G_1) - 2M_2(G_1)][HZ(G_2) - 2M_2(G_2)]$,
$\sum_{u\in V_1}\sum_{v\in V_2} [d^2(u)d(v) + d(u)d^2(v)] = 2m_2 M_1(G_1) + 2m_1 M_1(G_2)$,
$\sum_{u\in V_1}\sum_{v\in V_2} [d^3(u)d(v) + d(u)d^3(v)] = 2m_2[HZ(G_1) - 2M_2(G_1)] + 2m_1[HZ(G_2) - 2M_2(G_2)]$,
$\sum_{u\in V_1}\sum_{v\in V_2} [d^3(u)d^2(v) + d^2(u)d^3(v)] = [HZ(G_1) - 2M_2(G_1)]M_1(G_2) + [HZ(G_2) - 2M_2(G_2)]M_1(G_1)$,
$\sum_{u\in V_1}\sum_{v\in V_2} d^2(u)d^2(v) = M_1(G_1)M_1(G_2)$.

From (3.6.2), we have

$\sum_{(u,v)\epsilon V_1\times V_2} d^3(u,v) = n_2HZ(G_1) - 2n_2M_2(G_1) + n_1HZ(G_2) - 2n_1M_2(G_2) + [HZ(G_1) - 2M_2(G_1)][HZ(G_2) - 2M_2(G_2)] + 3[2m_2M_1(G_1) + 2m_1M_1(G_2)] + 6m_2[HZ(G_1) - 2M_2(G_1)] + 6m_1[HZ(G_2) - 2M_2(G_2)] + 3[HZ(G_1) - 2M_2(G_1)]M_1(G_2) + 3[HZ(G_2) - 2M_2(G_2)]M_1(G_1) + 6M_1(G_1)M_1(G_2)$ ……(3.6.3)

Again,

$\sum_{(a,x)\sim(b,y)} d(a,x)d(b,y) = \sum_{ab\epsilon E_1 \& xy\epsilon E_2}[d(a) + d(x) + d(a)d(x)][d(b) + d(y) + d(b)d(y)] + \sum_{a\epsilon V_1 \& xy\epsilon E_2}[d(a) + d(x) + d(a)d(x)][d(a) + d(y) + d(a)d(y)] + \sum_{ab\epsilon E_1 \& x\epsilon V_2}[d(a) + d(x) + d(a)d(x)][d(b) + d(x) + d(b)d(x)]$ ……(3.6.4)

Now,

$\sum_{ab\epsilon E_1 \& xy\epsilon E_2}[d(a) + d(x) + d(a)d(x)][d(b) + d(y) + d(b)d(y)] = \sum_{ab\epsilon E_1 \& xy\epsilon E_2}[d(a)d(b) + d(x)d(y)] + [d(a) + d(b)]d(x)d(y) + d(a)d(b)[d(x) + d(y)] + d(a)d(b)d(x)d(y) + [d(a)d(y) + d(b)d(x)]]$
$= 2[m_2M_2(G_1) + m_1M_2(G_2)] + 2[M_1(G_1)M_2(G_2) + M_1(G_2)M_2(G_1)] + 2M_2(G_1)M_2(G_2) + M_1(G_1)M_1(G_2)$ ……(3.6.5)

Similarly, we have

$\sum_{a\epsilon V_1 \& xy\epsilon E_2}[d(a) + d(x) + d(a)d(x)][d(a) + d(y) + d(a)d(y)] = m_2M_1(G_1) + [2m_1 + M_1(G_1)]M_1(G_2) + [n_1 + 4m_1 + M_1(G_1)]M_2(G_2)$ ……(3.6.6)

$\sum_{ab\epsilon E_1 \& x\epsilon V_2}[d(a) + d(x) + d(a)d(x)][d(b) + d(x) + d(b)d(x)] = m_1M_1(G_2) + [2m_2 + M_1(G_2)]M_1(G_1) + [n_2 + 4m_2 + M_1(G_2)]M_2(G_1)$ ……(3.6.7)

Using (3.6.5)-(3.6.7) in (3.6.4), we can obtain the following

$\sum_{(a,x)\sim(b,y)} d(a,x)d(b,y) = 3m_2M_1(G_1) + 3m_1M_1(G_2) + 3M_1(G_1)M_1(G_2) + 2M_2(G_1)M_2(G_2) + 6m_2M_2(G_1) + 3M_1(G_2)M_2(G_1) + n_2M_2(G_1) + 6m_1M_2(G_2) + 3M_1(G_1)M_2(G_2) + n_1M_2(G_2)$ ……(3.6.8)

Using (3.6.3) and (3.6.8) in (3.6.1), the desired result can be proved■

Theorem 3.7: Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with $|V_1| = n_1, |V_2| = n_2$ and $|E_1| = m_1, |E_2| = m_2$. Then

$HZ(G_1 \overline{\times} G_2) = HZ(G_1)HZ(G_2) + [3M_1(G_2) - 2M_2(G_2) + 6m_2 + n_2]HZ(G_1) - 2M_2(G_1)HZ(G_2) + 8M_2(G_1)M_2(G_2)$.

Proof: Let $a, b, u \epsilon V_1$, and $x, y, v \epsilon V_2$. We have,

$HZ(G \overline{\times} H) = \sum_{(a,x)\sim(b,y)}[d(a,x) + d(b,y)]^2$
$= \sum_{(a,x)\sim(b,y)}[d^2(a,x) + d^2(b,y)] + 2\sum_{(a,x)\sim(b,y)} d(a,x)d(b,y)$
$= \sum_{(u,v)\epsilon V_1\times V_2} d^3(u,v) + 2\sum_{(a,x)\sim(b,y)} d(a,x)d(b,y)$ ……(3.7.1)

Now,

$\sum_{(u,v)\epsilon V_1\times V_2} d^3(u,v) = \sum_{u\epsilon V_1}\sum_{v\epsilon V_2}[d(u)(1 + d(v))]^3$
$= \sum_{u\epsilon V_1} d^3(u)\sum_{v\epsilon V_2}[1 + d(v)]^3$
$= \sum_{u\epsilon V_1} d^3(u)\sum_{v\epsilon V_2}[d^3(v) + 3d^2(v) + 3d(v) + 1]$

Since,

$\sum_{u\epsilon V_1} d^3(u) = \sum_{pq\epsilon E_1}[d^2(p) + d^2(q)] = \sum_{pq\epsilon E_1}[d(p) + d(q)]^2 - 2d(p)d(q)$

$= HZ(G_1) - 2M_2(G_1)$

So,

$\sum_{(u,v)\epsilon V_1\times V_2} d^3(u,v) = [HZ(G_1) - 2M_2(G_1)][HZ(G_2) - 2M_2(G_2) + 3M_1(G_2) + 6m_2 + n_2]$ ……(3.7.2)

Again,

$\sum_{(a,x)\sim(b,y)} d(a,x)d(b,y)$

$= \sum_{a\sim b}\sum_{x=y\epsilon V_2} d(a)d(b)[d(x)+1]^2 + 2\sum_{a\sim b}\sum_{x\sim y} d(a)d(b)[d(x)d(y) + d(x) + d(y) + 1]$

$= \sum_{a\sim b} d(a)d(b)\sum_{x\epsilon V_2}[d^2(x) + 2d(x) + 1] + 2\sum_{a\sim b} d(a)d(b)\sum_{x\sim y}[d(x)d(y) + d(x) + d(y) + 1]$

$= M_2(G_1)[M_1(G_2) + 4m_2 + n_2] + 2M_2(G_1)[M_2(G_2) + M_1(G_2) + m_2]$ ……(3.7.3)

The desired result follows from equation (3.7.2), (3.7.3) and (3.7.1) ■

Theorem 3.8: Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with $|V_1| = n_1, |V_2| = n_2$ and $|E_1| = m_1, |E_2| = m_2$. Then

$\overline{HZ}(G_1 \otimes G_2) = (n_1n_2 - 2)M_1(G_1)M_1(G_2) + 16m_1^2m_2^2 - HZ(G_1)HZ(G_2) + 2[HZ(G_1)M_2(G_2) + HZ(G_2)M_2(G_1)] - 8M_2(G_1)M_2(G_2).$

Proof: Let $a, b \,\epsilon\, V(G_1)$, and $x, y \,\epsilon\, V(G_2)$. We have,

$\overline{HZ}(G_1\otimes G_2) + HZ(G_1\otimes G_2) = \sum_{(a,x)\sim(b,y)}[d(a,x) + d(b,y)]^2 + \sum_{(a,x)\nsim(b,y)}[d(a,x) + d(b,y)]^2$

$= \frac{1}{2}\left[\sum_{(a,x)\epsilon V(G_1\otimes G_2)}\sum_{(b,y)\epsilon V(G_1\otimes G_2)}[d(a,x) + d(b,y)]^2 - \sum_{(b,y)\epsilon V(G_1\otimes G_2)}[d(b,y) + d(b,y)]^2\right]$

$= \frac{1}{2}\left[\sum_{(a,x)\epsilon V(G_1\otimes G_2)}\sum_{(b,y)\epsilon V(G_1\otimes G_2)} d^2(a,x) + d^2(b,y) + 2d(a,x)d(b,y) - 4\sum_{(b,y)\epsilon V(G_1\otimes G_2)} d^2(b,y)\right]$

$= \frac{1}{2}[n_1n_2M_1(G_1\otimes G_2) + n_1n_2M_1(G_1\otimes G_2) + 2.4m_1m_2.4m_1m_2 - 4M_1(G_1\otimes G_2)]$

Therefore,

$\overline{HZ}(G_1\otimes G_2) = (n_1n_2 - 2)M_1(G_1)M_1(G_2) + 16m_1^2m_2^2 - HZ(G_1\otimes G_2)$

$\overline{HZ}(G_1\otimes G_2) = (n_1n_2 - 2)M_1(G_1)M_1(G_2) + 16m_1^2m_2^2 - HZ(G_1)HZ(G_2) + 2[HZ(G_1)M_2(G_2) + HZ(G_2)M_2(G_1)] - 8M_2(G_1)M_2(G_2)$

Hence the theorem■

Proposition 3.9: Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$be two graphs with $|V_1| = n_1, |V_2| = n_2$ and $|E_1| = m_1, |E_2| = m_2$. Then

$\overline{HZ}(G_1 \boxtimes G_2) = 4[m_1n_2 + m_2n_1 + 2m_1m_2]^2 - (n_1n_2 - 2)[(n_2 + 4m_2)M_1(G_1) + (n_1 + 4m_1)M_1(G_2) + M_1(G_1)M_1(G_2) + 8m_1m_2] - HZ(G_1 \boxtimes G_2).$

Proposition 3.10: Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with $|V_1| = n_1, |V_2| = n_2$ and $|E_1| = m_1, |E_2| = m_2$. Then

$\overline{HZ}(G_1 \overline{\times} G_2) = 4[m_1n_2 + 2m_1m_2]^2 - (n_1n_2 - 2)[M_1(G_1)(M_1(G_2) + 4m_2 + n_2)] - HZ(G_1 \overline{\times} G_2).$

Proposition 3.9-3.10 can be proved in a similar way, as adopted in *Theorem 3.8*. Also we can deduce these results using above theorems, *Theorem 3.6* and *Theorem 3.7* with *Proposition*

3.6 of [20]. As application we summarize the following formulae of Hyper-Zagreb index and it coindices for some known graph structures. The results in *Corollary 3.11* can be proved by direct calculation and by using *Theorem 3.8*, *Proposition 3.9-3.10*.

Corollary 3.11

1) $HZ(P_n \otimes P_m) = 128mn - 240n - 240m + 452$.
2) $HZ(P_n \boxtimes P_m) = 1024mn - 1602(m+n) + 2468$.
3) $HZ(King(m,n)) = 1024mn - 1602(m+n) + 2468$. The graph $P_n \boxtimes P_m$ is isomorphic to the $King(m,n)$ graph.
4) $HZ(P_n \overline{\times} P_m) = 432mn - 480m - 624n + 602$.
5) $HZ(C_n \otimes P_m) = 128mn - 240n$.
6) $HZ(C_n \boxtimes P_m) = 1024mn - 1602n$.
7) $HZ(C_n \overline{\times} P_m) = 432mn - 624n$.
8) $HZ(K_2 \otimes G(5,2)) = 1080$. The graph $K_2 \otimes G(5,2)$ is known as $Desargues'$ graph or $Desargues - levi$ graph. It is used to organize systems of stereoisomers of 5-ligand compounds.
9) $HZ(K_n \otimes K_n) = 2n^2(n-1)^3$.
10) $\overline{HZ}(P_n \otimes P_m) = 32m^2n^2 - 56mn^2 - 56m^2n - 60mn + 256n + 256m + 16n^2 + 16m^2 - 988$
11) $\overline{HZ}(C_n \overline{\times} P_m) = 16n^2 - 8mn^2 - 360mn + 544n$
12) $HZ(\overline{Rook(n,n)}) = 2n^2(n-1)^3$. The graph $K_n \otimes K_n$ is the complement of the graph $Rook(n,n)$.

4. CONCLUSION

In this paper, we derive expression of the Hyper-Zagreb index and its coindices of the tensor product, strong product and semi-strong product of graphs. The result of Hyper-Zagreb index of the tensor product of graphs is also generalized for *n* graphs.

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